Journal of King Saud University – Science (2013) 25, 175–178

King Saud University
Journal of King Saud University – Science
www.ksu.edu.sa
www.sciencedirect.com

ORIGINAL ARTICLE

# Challenge in the variational iteration method – A new approach to identification of the Lagrange multipliers

Guo-Cheng Wu *

College of Mathematics and Information Science, Neijiang Normal University, Neijiang 641112, China

Received 26 November 2012; accepted 8 December 2012
Available online 20 December 2012

KEYWORDS
Variational iteration method;
Adomian decomposition method;
Laplace transform;
Symbolic computation

**Abstract** The variational iteration method has been one of the most often used analytical methods in the past ten years. However, the success of the method mainly depends upon accurate identifications of the Lagrange multipliers. This study suggests a universal way to identify the multiplier which is a simple but effective approach by implementing Laplace transform. The Adomian series and the Pade technique are also employed to accelerate the convergence of the variational iteration algorithm. An example is given to elucidate the solution process and reliability of the solution.


## 1. Introduction

From the Lagrange multiplier method (Inokuti et al., 1978), the variational iteration method (VIM), which was first proposed by (He, 1998, 1999) and systematically elucidated in (He and Wu, 2007), has been worked out over a number of years by numerous authors. Since there is no need to handle nonlinear terms in an equation and it results in approximate solutions with high accuracies, the analytical method caught much attention in the past ten years, and it has matured into a relatively fledged theory thanks to the efforts of many researchers, notably (Abbasbandy, 2007; Noor and Mohyud-Din, 2008; Xu, 2009; Geng, 2010; Jafari and Khalique, 2012) to mention only a few. For a relatively comprehensive survey on the concepts, theory and applications of the method, readers are referred to review articles (He and Wu, 2007; He, 2012).

Recently, Laplace transform is adopted in some famous analytical methods (e.g., the Laplace Adomian decomposition method (LAPM) (Tsai and Chen, 2010; Zeng and Qin, 2012) and the homotopy perturbation method (Javidi and Raji, 2012)) to simplify the solution process and improve solution's accuracy. However, their applications mainly limit to the applications in differential equations with constant coefficients. Though the VIM can deal with such problems, the identification of the Lagrange multiplier is complex if not impossible. To solve the problem, the VIM is reconstructed and Laplace transform is adopted in simple and accurate identification of the multipliers.

## 2. A novel modification of the variational iteration method

We use the following general nonlinear equation to illustrate its basic idea of the VIM:

* Tel.: +86 0832 2341101.
E-mail address: wuguocheng2002@yahoo.com.cn.

1018-3647 

Peer review under responsibility of King Saud University.
doi:http://dx.doi.org/10.1016/j.jksus.2012.12.002

Production and hosting by Elsevier
ELSEVIER

$$\frac{d^m u}{dt^m} + R[u(t)] + N[u(t)] = g(t), u^{(k)}(0) = \frac{d^k u(0)}{dt^k},$$
$$k = 0, \ldots, m-1, \tag{1}$$

where $R$ is a linear operator, $N$ is a nonlinear operator, $g(t)$ is a given continuous function and $d^m u/dt^m$ is the term of the highest-order derivative.

The basic character of the method is to construct the following correction functional for Eq. (1)

$$u_{n+1} = u_n + \int_0^t \lambda(t,\tau)\left(\frac{d^m u_n}{d\tau^m} + R[u_n] + N[u_n] - g(\tau)\right)d\tau, \tag{2}$$

where $\lambda(t,\tau)$is a general Lagrange multiplier which can be identified optimally via the variational theory and $u_n$ is the $n$th order approximate solution.

The Lagrange multiplier can be identified as (He and Wu, 2007)

$$\lambda(t,\tau) = \frac{(-1)^m(\tau - t)^{m-1}}{(m-1)!}. \tag{3}$$

Inspired by Tsai and Chen's method (Tsai and Chen, 2010), if we take Laplace transform on both sides of (1), the linear part with constant coefficients is then transferred into an algebraic one so that we can identify the Lagrange multiplier in a more straightforward way.

Now considering the system (1) with variable coefficients and assuming the linear term as

$$R[u(t)] = \sum_{i=0}^{m-1}(a_i u^{(i)} + b_i(t)u^{(i)}), \tag{4}$$

where $u^{(i)} = u^{(i)}\ (t)$, $a_i$ is a constant and $b_i(t)$ is a variable coefficient, a novel modified VIM is given as follows:

I. Take the Laplace transform on (1), then the iteration formula becomes

$$U_{n+1}(s) \ = U_n(s) + \lambda\left[s^m U_n(s) - \sum_{i=0}^{m-1} u^{(i)}(0)s^{m-i-1} + L\left(\sum_{i=0}^{m-1} a_i u_n^{(i)}\right)\right.$$
$$\left. + L\left(\sum_{i=0}^{m-1} b_i(t)u_n^{(i)}\right) + L(N[u_n] - g(t))\right] \tag{5}$$

where $U(s)$ is Laplace transform of $u(t)$ and $s$ is a complex variable.

II. Consider the terms $L\left(\sum_{i=0}^{m-1} b_i(t)u^{(i)}\right)$ and $L(N[u_n])$ as restricted variations. Make Eq. (5) stationary with respect to $U_n$

$$\delta U_{n+1}(s) \ = \delta U_n(s) + \lambda\left[s^m \delta U_n(s) + \sum_{i=0}^{m-1} a_i s^i \delta U_n(s)\right]. \tag{6}$$

From Eq. (6), we can determine the Lagrange multiplier as follows

$$\lambda(s) = -\frac{1}{\sum_{i=0}^{m} a_i s^i},\ a_m = 1.$$

III. The variational iteration formula is obtained through the inverse Laplace transform $L^{-1}$:

$$u_{n+1}(t) \ = u_n(t) + L^{-1}\left[\lambda(s)\left[s^m U_n(s) - \sum_{k=0}^{m-1} u^{(k)}(0)s^{m-k-1}\right.\right.$$
$$\left.\left. + L\left(\sum_{i=0}^{m-1} a_i u_n^{(i)}\right) + L\left(\sum_{i=0}^{m-1} b_i(t)u_n^{(i)}\right) + L(N[u_n] - g(t))\right]\right]$$
$$= u_0 + L^{-1}\left[\lambda(s)L(\sum_{i=0}^{m-1} b_i(t)u_n^{(i)} + N[u_n])\right],$$

where the initial iteration value can be determined as

$$u_0 = L^{-1}\left[\lambda(s)\left[-\sum_{k=0}^{m-1} u^{(k)}(0)s^{m-k-1} - \left(\sum_{i=0}^{m-1} a_i \sum_{k=0}^{i-1} u^{(k)}(0)s^{i-1-k}\right)\right.\right.$$
$$\left.\left. - L[g(t)]\right]\right]$$

IV. Let $u_n = \sum_{j=0}^{n} v_j$ and apply the Adomian decomposition method (ADM) (Adomian, 1994) to expand the term $N[u]$ as $\sum_{j=0}^{\infty} A_j$. Then the iteration formula reads

$$\begin{cases} v_{j+1} \ = L^{-1}\left[\lambda(s)L(\sum_{i=0}^{m-1} b_i(t)v_j^{(i)} + A_j)\right], \\ v_0 = u_0, \end{cases}$$

where $A_j$ is the famous Adomian decomposition series. A detailed review of the ADM can be found (See Duan et al., 2012).

V. Employ the Pade-technique (Baker and Graves-Morris, 1996) to accelerate the convergence of $u_n$.

Remarks:

If $a_i$ is a constant, the presented algorithm reduces to Tsay and Chen's LAPM. On the other hand, since the VIM allows the terms with variable coefficients as restricted variations, as a result, the modified VIM here is more flexible and general here.

This idea can also be extended to the fractional calculus of variations (Baleanu and Trujillo, 2008,2010). For the VIM in the fractional differential equations, the Lagrange multipliers can be determined readily from Laplace transform. Readers who feel interested in the method are referred to the recent development (Wu, 2012a, b, c).

Consider the following differential equations with a variable coefficient as an example

$$\frac{d^2 u}{dt^2} = 1 + (1 + 2t)u + u^3,\ u(0) = 0, \quad u'(0) = 1 \tag{7}$$

Take Laplace transform on both sides of (7)

$$s^2 U(s) - u'(0) - u(0)s = \frac{1}{s} + U(s) + L[2tu] + L[u^3] \tag{8}$$

Construct the variational iteration formula for (8) as

$$U_{n+1}(s) \ = U_n(s) + \lambda[s^2 U_n(s) - u'(0) - u(0)s - \frac{1}{s} - U_n(s)$$
$$- L[2tu_n] - L[u_n^3]] \tag{9}$$

Considering the terms $L[2tu]$ and $L[u^3]$ are restricted variations, take the classical variation operator on both sides of (9)

$$\delta U_{n+1}(s) \ = \delta U_n(s) + \lambda(s^2 - 1)\delta U_n(s)$$

As a result, a Lagrange multiplier can be determined as

$$\lambda = -\frac{1}{s^2 - 1} \tag{10}$$

Substituting (10) into (9), one can obtain

$$U_{n+1}(s) = U_n(s) + \lambda\left[s^2U_n(s) - u'(0) - u(0)s - \frac{1}{s} - U_n(s)\right.$$
$$\left. - L[2tu_n] - L[u_n^3]\right] = \frac{1}{s^2-1}\left[u'(0) + u(0)s + \frac{1}{s}\right]$$
$$+ \frac{1}{s^2-1}\left(L[2tu_n] + L[u_n^3]\right) \tag{11}$$

and use the inverse Laplace transform $L^{-1}$,

$$u_{n+1}(t) = u_0 + L^{-1}\left[\frac{1}{s^2-1}L[2tu_n + u_n^3]\right]$$

where the initial iteration value is

$$u_0 = L^{-1}\left[\frac{1}{s^2-1}\left[u'(0) + u(0)s + \frac{1}{s}\right]\right] = e^t - 1 \tag{12}$$

Furthermore, linearizing the term $u^2$ implemented by the Adomian series, let $u_n = \sum_{i=0}^{n} v_i$ and the nonlinear term $u^3$ can be approximately expanded as the Adomian polynomials. As a result, we can derive the following iteration formula

$$\begin{cases} v_{i+1} = L^{-1}\left[\frac{1}{s^2-1}\left(L[2tv_i] + L[A_i]\right)\right], \\ v_0 = u_0 \end{cases}$$

where the Adomian series of the term $v^3$ reads

$$\begin{cases} A_0 = v_0^3, \\ A_1 = 3v_0^2v_1, \\ A_2 = 3v_0v_1^2 + 3v_0^2v_2, \\ \quad\vdots \end{cases}$$

With symbolic computation, we can derive the following approximate solutions

$$\begin{aligned} &u_0 = e^t - 1, \\ &u_1 = e^t - \tfrac{11}{12}e^{-t} - \tfrac{1}{3}e^{2t} - 2t - \tfrac{e^t}{4}(2t^2 - 1 - 6t), \\ &\vdots \end{aligned} \tag{13}$$

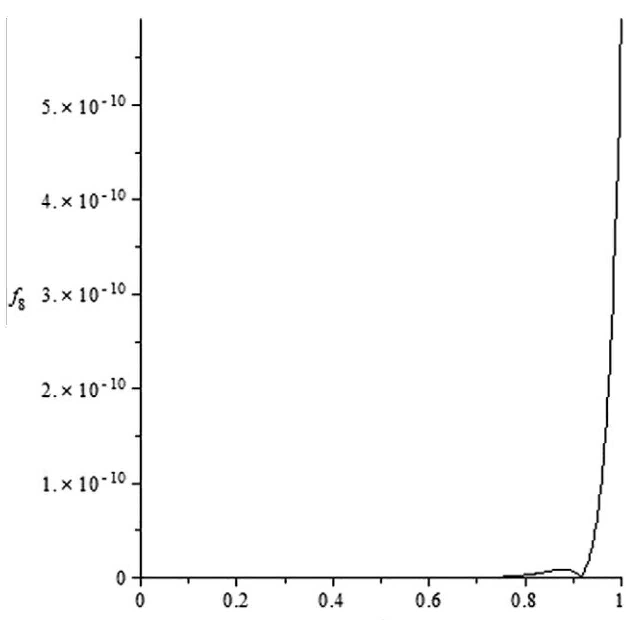


**Figure 1** Curve of the residual value $f_8$.

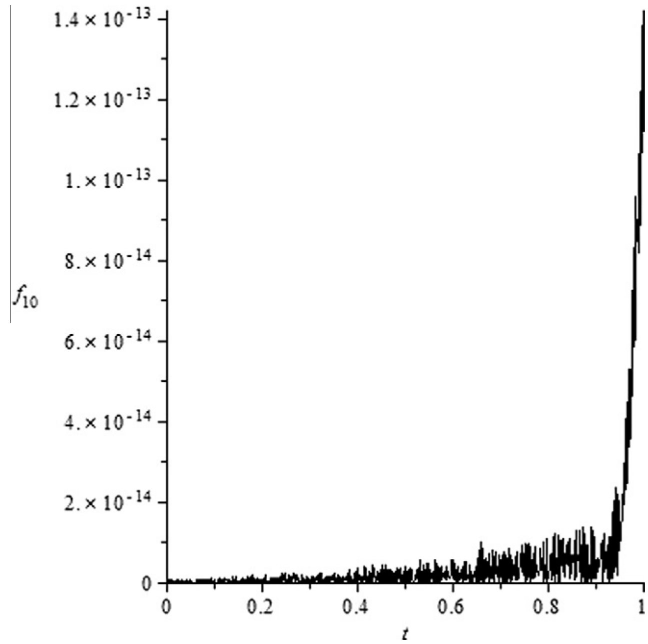


**Figure 2** Curve of the residual value $f_{10}$.

Now apply the Pade technique to $u_8$ and $u_{10}$ and denote the results by $u_8\left[\frac{10}{10}\right]$ and $u_{10}\left[\frac{16}{16}\right]$. Define the residual value function as

$$f_n = \left|\frac{d^2(u_n[])}{dt^2} - 1 - (1+2t)(u_n[]) - (u_n[])^3\right| \tag{14}$$

where $u_{n_{[]}}$ is the $n$-th order approximate solution $u_n$ after using the Pade technique. Now substituting $u_8\left[\frac{10}{10}\right]$ and $u_{10}\left[\frac{16}{16}\right]$ into (14), the residual value function $f_n$ in the interval $[0,1]$ is illustrated in Figs. 1 and 2, respectively.

On the other hand, we also can have the following variational iteration formula for (7)

$$\begin{cases} u_{n+1}(t) = u_0 + L^{-1}[\lambda(s)L[(1+2t)u_n + u_n^3]], \\ \lambda(s) = -1/s^2, \\ u_0 = t + t^2/2 \end{cases}$$

if only the term $\frac{d^2u}{dt^2}$ is included in the calculus of variations.

## 4. Conclusions

The VIM has been extensively used for solving various differential equations. The critical step of the method is to identify the Lagrange multipliers through calculus of variations. In this study, the VIM is improved with Laplace transform and the Adomian series. Compared with the classical VIM, the modified version method has following merits: (a) the Lagrange multipliers can be readily obtained in a more straightforward way; (b) the initial iteration value can be determined universally; (c) The method is also powerful to solving differential equations with variable coefficients. The error analysis and higher order approximate solutions of the nonlinear oscillator illustrate the method's efficiencies and high accuracies.

## References

Abbasbandy, S., 2007. A new application of He's variational iteration method for quadratic Riccati differential equation by using Adomian's polynomials. J. Comput. Appl. Math. 207 (1), 59–63.

Adomian, G., 1994. Solving Frontier Problems of Physics: The Decomposition Method. Kluwer Academic Publishers.

Baker, G.A., Graves-Morris, P., 1996. Padé Approximants. Cambridge University Press.

Baleanu, D., Trujillo, J.J., 2008. On exact solutions of a class of fractional Euler–Lagrange equations. Nonlinear Dyn. 52 (4), 331–335.

Baleanu, D., Trujillo, J.J., 2010. A new method of finding the fractional Euler–Lagrange and Hamilton equations within Caputo fractional derivatives. Commun. Nonlinear Sci. Numer. Simul. 15 (5), 1111–1115.

Duan, J.S., Rach, R., Baleanu, D., Wazwaz, A.M., 2012. A review of the Adomian decomposition method and its applications to fractional differential equations. Commun. Fract. Calc. 3 (2), 73–99.

Geng, F., 2010. A modified variational iteration method for solving Riccati differential equations. Comput. Math. Appl. 60 (7), 1868–1872.

He, J.H., 1998. Approximate analytical solution for seepage flow with fractional derivatives in porous media. Comput. Methods Appl. Mech. Eng. 167 (1–2), 57–68.

He, J.H., 1999. Variational iteration method – a kind of nonlinear analytical technique: some examples. Int. J. Nonlinear Mech. 34 (4), 699–708.

He, J.H., Wu, X.H., 2007. Variational iteration method: new development and applications. Comput. Math. Appl. 54 (7–8), 881–894.

He, J.H., 2012. Asymptotic methods for solitary solutions and compactons. Abstr. Appl. Anal. 2012, 916793.

Inokuti, M., Sekine, H., Mura, T., 1978. General use of the Lagrange multiplier in nonlinear mathematical physics. In: Nemat-Nasser, S. (Ed.), Variational Methods in the Mechanics of Solids. Pregman Press, NY, pp. 156–162.

Jafari, H., Khalique, C., 2012. Homotopy perturbation and variational iteration methods for solving fuzzy differential equations. Commun. Fract. Calc. 3 (1), 38–48.

Javidi, M., Raji, M.A., 2012. Combination of Laplace transform and homotopy perturbation method to solve the parabolic partial differential equations. Commun. Fract. Calc. 3 (1), 10–19.

Noor, M.A., Mohyud-Din, S.T., 2008. Variational iteration method for solving higher-order nonlinear boundary value problems using He's polynomials. Int. J. Nonlinear Sci. Numer. Simul. 9 (2), 141–156.

Tsai, P.Y., Chen, C.K., 2010. An approximate analytic solution of the nonlinear Riccati differential equation. J. Franklin Inst. 347 (10), 1850–1862.

Wu, G.C., 2012a. Laplace transform overcoming principle drawbacks in application of the variational iteration method to fractional heat equations. Therm. Sci. 16 (4), 1257–1261.

Wu, G.C., 2012b. Variational iteration method for solving the time-fractional diffusion equations in porous medium. Chin. Phys. B 21 (12), 120504.

Wu, G.C., 2012c. Applications of the variational iteration method to fractional diffusion equations: local versus non-local ones. Int. Rev. Chem. Eng. 4 (5), 505–510.

Xu, Lan., 2009. The variational iteration method for fourth order boundary value problems. Chaos, Solitons Fractals 39 (3), 1386–1394.

Zeng, D.Q., Qin, Y.M., 2012. The Laplace–Adomian–Pade technique for the seepage flows with the Riemann–Liouville derivatives. Commun. Fract. Calc. 3 (1), 26–29.